Definitions of mass in special relativity

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Traditionally, most undergraduate textbooks on special relativity (e.g. Rosser 1967, French 1968) follow many of the specialized monographs (e.g. Møller 1952, Aharoni 1965) in using the concept of the relativistic mass $m_r(v)$ of a particle, equal to $\gamma(v)m_0$, where m_0 is the rest mass of the particle and $\gamma(v)$ is equal to $(1 - v^2/c^2)^{-1/2}$. In terms of this quantity, the momentum and energy of the particle are given by $m_r v$ and $m_r c^2$.

Recently, some textbooks, in particular the popular and stimulating one by Taylor and Wheeler (1963) (to be referred to as TW) have not used the relativistic mass, but have merely defined momentum and energy as $\gamma m_0 v$ and $\gamma m_0 c^2$. We shall briefly compare the two approaches, and then proceed to the main point of the article, which is to discuss a particular problem within the TW approach.

Discussion of the approaches

The relativistic mass approach, as mentioned above, has become traditional, though it is interesting that Einstein did not use it in his original paper or in his popular account of relativity (Einstein 1905, 1956). The concept of relativistic mass is obviously a convenient simplification in the definitions of momentum and energy. It is also useful in the expression for the total momentum of a system, and its use enables a centre of relativistic mass to be defined and used. The fact that it is simply related to energy means that it has many of the important properties of energy; it is conserved and the relativistic masses of a number of particles sum to give the total relativistic mass of the system of particles.

But the relativistic mass cannot be regarded as a fundamentally important quantity. Precisely because it is essentially identical to energy, it adds little to an understanding of the situation that is not known from energy considerations. That it is of limited significance is clear from the fact that, for instance, it cannot be substituted for rest mass in the classical expression of Newton's second law, or (less significantly) in the formula for kinetic energy, to give the relativistic expressions. It is necessary, instead, to return to the basic definitions of energy and momentum.

The most important criticism of the relativistic mass approach is that it gives the impression that the effects of relativity are due to 'something happening' to the particle, whereas they are of course due to the properties of space–time. Indeed, from the space–time point of view the relativistic mass immediately loses much of its significance, as it is not an invariant under Lorentz transformations. The TW approach, on the other hand, emphasizes the space–time approach as it focuses attention on the concept of the momentum–energy four-vector, the scalar product of which is a Lorentz invariant and essentially equal to the square of the rest mass:

$$p^2 - E^2/c^2 = -m_0^2c^2. \qquad (1)$$

While the more conventional equation equating E to $m_r c^2$ demonstrates the connection between energy and mass, equation (1) has the advantage of showing that the energy of a particle depends both on its rest mass and on its momentum.

The difficulties mentioned above with the relativistic mass approach do not occur in the TW approach, and it can be seen that a good case can be made for it to be adopted generally. Personal experience shows that with a little familiarity with the TW approach, the relativistic mass concept appears rather artificial.

Mass of a system

Attention is obviously directed by equation (1) towards the rest mass, but there is one aspect of the equation, fully accepted by TW, which has been found to confuse students and to lead them into error. (It will be shown later that this difficulty is not avoided in the relativistic mass approach.)

The problem is the application of equation (1) to a system of particles which are not all moving at the same velocity. We shall take the simplest example, that of two particles of equal rest mass m_0, travelling towards each other with equal speeds v in a particular frame. (Note that I use the term 'rest mass' to avoid any confusion, although in the context of the TW approach the simple term 'mass' would imply 'rest mass' as no other type of mass is considered.) In the frame we are considering, the momentum of the system is found, of course, to be zero, while the energy is equal to $2\gamma m_0 c^2$. Application of equation (1) yields the result that the rest mass of the system is equal to $2\gamma m_0$, i.e. it is greater than the sum of the rest masses of the individual particles. (It is obviously a general result that, in the frame in which there is zero momentum, the rest mass of the system is equal, apart from factors of c, to the energy of the system.)

There is nothing incorrect physically in this reasoning; careful use of the ideas involved can lead to no errors in results. It must be admitted, though, that the idea of the rest mass of a system of particles being greater than the sum of the rest masses of the individual particles is very difficult to accept. In the course of obtaining a full grasp of relativistic physics, a student has to come to terms with many ideas completely foreign to his everyday experience, but this does not imply that another should be added if it is not necessary. There is an additional disadvantage here in that the student may not meet the difficulty in an introductory discussion of relativistic mechanics, which would be concerned primarily with single particles, and so he could be unprepared for problems concerning systems of particles, and suffer major confusion.

Incidentally, in the relativistic mass approach as well, there is a complication when the properties of a system of particles are considered. It is true that the total momentum P and the total energy E are related to the sum of the relativistic masses M_r in the same way as the corresponding quantities are related for a particle. The velocity to be used in this case is the velocity v_c of the centre of relativistic mass. But there is no way of obtaining the relativistic mass of the system, apart from adding the individual relativistic masses. In other words, M_r cannot be expressed as $\Sigma m_{i0}\,(1 - v_i^2/c^2)^{-1/2}$, or indeed by any similar expression with a different function of the rest masses in the numerator. Thus at some stage in this method it is necessary to consider individual particles; the formulation cannot be applied solely with the use of quantities for the system. There is some similarity to the situation with the TW approach.

Inelastic collisions

We shall now return to the TW approach and see how this definition of the rest mass of a system may lead to lack of clarity of thought, and even error. We will discuss an inelastic collision with coefficient of restitution zero, between the two particles mentioned above. There are two cases. The first is where the particles are elementary, and a third elementary particle is produced in the collision. Its rest mass must equal $2\gamma m_0$ and this of course is the true rest mass of the particle, provided it is formed in its ground state. The final particle is not in any sense to be thought of as 'composed of' the particles which gave rise to its creation. (In practice the reaction would be difficult to induce; the slightest departure from v of the velocity of the particles in the zero momentum frame would mean that energy and momentum could not both be conserved.)

This analysis is applied by TW (p121) to the case of two macroscopic particles, such as balls of putty, coalescing on collision. This is ascribing a rest mass of $2\gamma m_0$ to the composite ball of putty immediately after collision. In effect, the expression for the rest mass of the system includes the kinetic energy, that is to say the energy of rotation of the balls about their centre of mass, and the energy involved in the excitation of internal degrees of freedom of the system. In time the rotational energy will be lost through friction, and the internal energy will be transferred as heat, and possibly light and sound. Finally we are left with a ball of putty of rest mass $2m_0$.

This description appears unhelpful and liable to confuse. The very real differences between the two types of inelastic collisions are obscured because the rest mass directly after collision is made to be the same in each case. It is also physically unconvincing to talk of 'the collision' as if it were a single event. In practice there will be numerous interactions between atoms in the balls of putty occurring over a finite time, and even during this period of time energy will be lost from the balls as heat. Thus to describe a state after the collision in which the rest mass is $2\gamma m_0$ appears to be actually incorrect.

The point of these remarks is that when the TW definition of the rest mass of the system is used, attention is directed towards unimportant matters such as the flow of energy from the system to its surroundings, and diverted from the real issues, the conservation of energy and momentum and the possibility of the sum of the rest masses of the particles of the system not remaining constant.

Definition of 'single particle'

Thus, while I would, as stated above, commend the main structure of the TW approach, it would be preferable to use equation (1) for 'single particles'. The term 'single particle' obviously needs some clarification, as many of the entities to be considered are eminently not elementary, and even the majority of the so called elementary particles are thought to have internal structure. The suitable definition is that a number of particles may be considered to be a 'single particle' if their motion is the resultant of (i) a translational motion common to all the particles, and (ii) motion of internal degrees of freedom of the 'single particle' with the energy appropriate to the temperature of the surroundings. In the zero momentum frame, (i) is eliminated. To explain the point of this definition we shall consider further the two classes of inelastic collision described above.

In the first class, that between two elementary particles, we would obviously not consider treating both particles together as a 'single particle', because the motion cannot be considered thermal at any temperature. Each individual elementary particle, however, is to be considered a 'single particle', even if internal degrees of freedom are present, because the amount of energy attributable to these degrees of freedom is that appropriate to that temperature.

It is not necessary to discuss the relation between the rest mass of the 'single particle' and those of its constituents, but in fact we have:

$$m_0 = \sum_i m_{i0} - E_b/c^2, \tag{2}$$

where E_b is the binding energy of the particle calculated in the zero momentum frame. This is, in fact, an extension to the use of equation (1) for the 'single particle' as that equation states that m_0 equals the sum of the individual rest masses added to the kinetic energy (divided by c^2) calculated in the zero momentum frame. To this we must add the (negative) potential energy (divided by c^2) to give equation (2); equation (1) only is meant to apply to noninteracting particles.

The particle produced in the collision may also be considered to be a 'single particle', at least provided it is formed in its ground state. If this is so, the internal degrees of freedom are in equilibrium with the surroundings, and we use equation (1) to obtain a value for the rest mass of $2\gamma m_0$. If the particle is formed in an excited state, we do not consider it to be a 'single particle' until it decays to its ground state (or in other words until its internal degrees of freedom come into equilibrium with the surroundings). Thus if application of equation (1) to the particle in its ground state yields the value m_1 (less than $2\gamma m_0$), we define its mass to be m_1 even when it is in its excited state. The energy of excitation is then $(2\gamma m_0 - m_1)/c^2$.

Before the collision, the rest mass is defined as being equal to the sum of the rest masses of the two 'single particles'. Thus the initial rest mass is $2m_0$, and the final rest mass is $2\gamma m_0$, provided the particle is formed in its ground state. So in the collision, kinetic energy is transformed into rest mass.

When we turn to the second class of inelastic collision, that which involves balls of putty, we note that in this case there are obviously internal degrees of freedom for each ball of putty. Nevertheless, because these degrees of freedom are again in equilibrium at the temperature of the surroundings, it is possible to consider them as 'single particles' and equation (1) may therefore be applied. The same applies to the final ball of putty once it has reached equilibrium with its surroundings. This does not apply during the period following the collision, however, when the composite ball is not in equilibrium with its surroundings. We do not, therefore, use equation (1), but define the rest mass of the ball at this stage to be equal to the final rest mass when it has reached equilibrium.

This final rest mass is given by an equation similar to equation (2), but in this case the binding energy is entirely negligible, and so the final rest mass is equal to the sum of the initial rest masses. Thus the rest mass of the system is constant throughout in this case, energy being transformed from translational energy to rotational and internal energy of the composite ball of putty, and finally to heat.

To sum up, we do not use equation (1) if the system consists of noninteracting particles with differing translational motions, or for particles combined to form a single entity, when the internal degrees of freedom are not in equilibrium with the surroundings. In the first case, the rest mass of the system is to be defined as the sum of the rest masses of the particles. In the second case, the rest mass is defined to have the value it will have once equilibrium is reached, which is essentially equation (2). The binding energy in this equation may vary from the putty case, where it is negligible, to the case of a nucleon composed (possibly) of quarks, where it may be very much larger than the rest mass of the nucleon multiplied by c^2.

Conclusion

It has been shown that the TW definition of the rest mass of a system can lead to confusion. Alternative definitions of rest mass have been proposed, which agree closer with our expectations.

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